

POLYTOPAL AFFINE SEMIGROUPS WITH HOLES DEEP INSIDE

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ABSTRACT. Given a non-negative integer k , we construct a lattice polytope P with the following property: The affine semigroup Q_P associated to P is not normal, and every element $q \in \overline{Q}_P \setminus Q_P$ has lattice distance at least k above every facet of Q_P .

1. INTRODUCTION

Let $P \subset \mathbb{R}^n$ be a lattice polytope, i.e. a polytope whose vertices have integer coordinates. We consider the affine semigroup $Q = Q_P \subset \mathbb{Z}^{n+1}$ generated by the points $\{(p, 1) \in \mathbb{Z}^{n+1} \mid p \in P \cap \mathbb{Z}^n\}$. Let $\mathbb{Z}Q \subset \mathbb{Z}^{n+1}$ be the group generated by the elements of Q . We write $\overline{Q} = \mathbb{Z}Q \cap \mathbb{R}_{\geq 0}Q$ for the normalization of Q . Equivalently, \overline{Q} contains all elements of $\mathbb{Z}Q$, such that a positive integral multiple is contained in Q . Then P resp. Q_P are called *normal* if $Q_P = \overline{Q}_P$. It is a much studied question to characterize normal polytopes. See [2] for background information on affine semigroups and normal polytopes. There are results that suggest that the normality of P is somehow determined by the “boundary” of P , see for example [3]. Therefore, it seems natural to ask if it is enough to consider normality “near the boundary”. To make this precise, we give some definitions. We call an element $q \in \overline{Q} \setminus Q$ a *hole* in Q . The holes come in families of different dimension, cf. [4]. For a facet F of Q_P , let $\sigma_F : \mathbb{Z}Q_P \rightarrow \mathbb{Z}$ be the *lattice height* above F , i.e. the linear form with $\sigma_F(F) = 0$ and $\sigma_F(\overline{Q}) = \mathbb{Z}_{\geq 0}$, cf. [2, Remark 1.72]. It is enough to consider elements of lattice height at most 1 in \overline{Q} to detect families of holes of dimension d , see [2, Exercise 4.15]. The main result of the present note is that this observation does not generalize to higher codimension.

Theorem 1.1. *For every natural number $k \in \mathbb{N}$, there exists a 3-simplex $P = P(k)$, such that the polytopal affine semigroup Q_P is not normal, and every hole $q \in \overline{Q}_P \setminus Q_P$ has a lattice height of at least k above each facet of Q_P .*

In other words, there are polytopes P , such all holes of the semigroup Q_P are “deep inside”. So it is not sufficient to look for holes near the boundary. Note that this result is trivial if one considers more general affine semigroups that are not polytopal. One may just take a big normal polytope P and remove a point from its far interior to obtain a homogeneous affine semigroups with the desired property.

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2. RECTANGULAR SIMPLICES

We will construct the simplices in [Theorem 1.1](#) as a special case of the *rectangular simplices* introduced in [\[1\]](#). In this section we recall the construction. Let $\mathbf{e}_i \in \mathbb{R}^{n+1}$ denote the i -th unit vector. Let $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n)$ be a vector of positive integers. We consider the simplex $\Delta = \Delta(\boldsymbol{\lambda}) \subset \mathbb{R}^{n+1}$ with vertices

$$\begin{aligned} \mathbf{v}_0 &:= (0, 0, \dots, 0, 1) &= \mathbf{e}_{n+1}, \\ \mathbf{v}_1 &:= (\lambda_1, 0, \dots, 0, 1) &= \lambda_1 \mathbf{e}_1 + \mathbf{e}_{n+1}, \\ \mathbf{v}_2 &:= (0, \lambda_2, \dots, 0, 1) &= \lambda_2 \mathbf{e}_2 + \mathbf{e}_{n+1}, \\ &\vdots &\vdots \\ \mathbf{v}_n &:= (0, \dots, 0, \lambda_n, 1) &= \lambda_n \mathbf{e}_n + \mathbf{e}_{n+1}. \end{aligned}$$

Write $Q = Q(\boldsymbol{\lambda})$ for the affine semigroup generated associated to $\Delta(\boldsymbol{\lambda})$. Note that $\mathbb{Z}Q = \mathbb{Z}^{n+1}$, because $\mathbf{e}_{n+1}, \mathbf{e}_1 + \mathbf{e}_{n+1}, \dots, \mathbf{e}_n + \mathbf{e}_{n+1} \in Q$. There are two kinds of facets of Q :

- The coordinate hyperplanes are facets of Q . We denote the facet defined by the i -th coordinate hyperplane by F_i . The lattice height σ_i above F_i is given by the i -th coordinate of a point $q \in \mathbb{Z}Q$.
- There is one “skew” facet spanned by the vertices $\lambda_i \mathbf{e}_i + \mathbf{e}_{n+1}$, $1 \leq i \leq d$. Let us denote this facet by $F_{\boldsymbol{\lambda}}$. The lattice height above this facet is given by the linear form

$$\sigma_{\boldsymbol{\lambda}}(z) := Lz_{n+1} - \sum_{i=1}^n \frac{L}{\lambda_i} z_i,$$

where $L := \text{lcm}(\lambda_1, \dots, \lambda_n)$.

3. REDUCTION TO THE SKEW FACET

In this section, we prove the following result that allows us to restrict our attention to the facet $F_{\boldsymbol{\lambda}}$.

Proposition 3.1. *Let k be a positive integer. Assume that $Q(\boldsymbol{\lambda})$ is not normal and every hole has lattice height at least k above $F_{\boldsymbol{\lambda}}$. Assume further that $Q(\boldsymbol{\lambda})$ has no holes in its boundary. Then there exists a $\boldsymbol{\lambda}'$ such that $Q(\boldsymbol{\lambda}')$ is not normal and its holes have lattice height at least k above every facet.*

The idea is taken from [\[1, Theorem 1.6\]](#). For a fixed index $1 \leq i \leq n$, set $\ell = \text{lcm}(\lambda_1, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_n)$. We define $\boldsymbol{\lambda}' = (\lambda'_1, \dots, \lambda'_n)$ by

$$\lambda'_j = \begin{cases} \lambda_j & \text{if } j \neq i; \\ \lambda_j + \ell & \text{if } j = i. \end{cases}$$

Theorem 1.6 of [\[1\]](#) states that in this situation $Q(\boldsymbol{\lambda})$ is normal if and only if $Q(\boldsymbol{\lambda}')$ is normal. We modify the argument given in [\[1\]](#) to obtain the following result.

Lemma 3.2. *Use the notation as above. Assume that $Q(\boldsymbol{\lambda})$ has no holes in its boundary. Then there is a bijective linear map $\alpha : \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}^{n+1}$, such that the preimage of every hole in $Q(\boldsymbol{\lambda}')$ is a hole in $Q(\boldsymbol{\lambda})$ (i.e. α is surjective on holes). Moreover, α strictly increases the lattice height of every hole above the facet F_i , and it preserves all other lattice heights.*

We can iterate this construction to increase the lattice height of the holes above every facet except F_{λ} . This proves [Proposition 3.1](#). The map α is taken from the proof of Theorem 1.6 in [1]; we give its definition below. For the proof of [Lemma 3.2](#), we need the following lemma.

Lemma 3.3. *Let $z \in Q(\lambda)$, $\tilde{z} \in \mathbb{Z}^{n+1}$ with $0 \leq \tilde{z}_i \leq z_i$ for $1 \leq i \leq n$ and $\tilde{z}_{n+1} = z_{n+1}$. Then $\tilde{z} \in Q(\lambda)$.*

Proof. We first note that the statement holds if $z_{n+1} = 1$. This follows from the definition of the simplex $\Delta(\lambda)$. In general, z can be written as a sum of elements of degree 1. For each summand, we may decrease its components without leaving $Q(\lambda)$. This way, we obtain a representation of \tilde{z} as a sum of degree 1 elements. Hence, $\tilde{z} \in Q(\lambda)$. \square

Proof of Lemma 3.2. Set $L = \text{lcm}(\lambda_1, \dots, \lambda_n)$ and $L' = \text{lcm}(\lambda'_1, \dots, \lambda'_n)$. Recall that

$$\sigma_{\lambda}(z) = Lz_{n+1} - \sum_{i=1}^n \frac{L}{\lambda_i} z_i$$

and analogously for λ' . We consider the linear form

$$\beta(z) := \frac{\ell}{L} \left(\sigma_{\lambda}(z) + \frac{L}{\lambda_i} \sigma_i(z) \right) = \ell z_{n+1} - \sum_{\substack{j=1 \\ j \neq i}}^n \frac{\ell}{\lambda_j} z_j.$$

defined on \mathbb{Z}^{n+1} . Note that β takes non-negative integer values on $Q(\lambda)$. The map α mentioned above can be defined by $\alpha(z) := z + \beta(z)\mathbf{e}_i$.

One can directly verify that $\sigma_{\lambda'}(\alpha(z)) = \sigma_{\lambda}(z)$ for every $z \in \mathbb{Z}^{n+1}$. It follows that α preserves the height above every facet except F_i . Since $Q(\lambda)$ has no holes in its boundary, every hole z has $\sigma_{\lambda}(z) > 0$, so the height above F_i is strictly increased.

It remains to show that α is surjective on holes. As a preparation, we show that $\alpha(Q(\lambda)) \subset Q(\lambda')$. We first note that it follows from the discussion above that $\alpha(\overline{Q}(\lambda)) \subset \overline{Q}(\lambda')$. Next, consider an element $w \in Q(\lambda)$. It can be written as a sum of elements of degree 1. Since α preserves the degree, this yields a representation of its image $\alpha(w)$ as a sum of degree 1 elements of $\overline{Q}(\lambda')$. But $Q(\lambda')$ coincides with $\overline{Q}(\lambda')$ in degree 1, hence $\alpha(w) \in Q(\lambda')$.

Let $z' \in \overline{Q}(\lambda') \setminus Q(\lambda')$ be a hole and set $z := \alpha^{-1}(z')$. We need to show that z is a hole of $Q(\lambda)$. It is immediate that $z \notin Q(\lambda)$, because otherwise $z' = \alpha(z) \in Q(\lambda')$. It remains to show that $z \in \overline{Q}(\lambda)$, so assume the contrary. Then $z_i < 0$, or equivalently, $z'_i < \beta(z)$. Let $\tilde{z}' := z' + (\beta(z) - z'_i)\mathbf{e}_i$. The linear form β does not depend on z_i nor on λ_i , therefore

$$\beta(z') = \beta(z) = \frac{\ell}{L'} \left(\sigma_{\lambda'}(z') + \frac{L'}{\lambda'_i} \sigma_i(z') \right)$$

Using this, we compute

$$\begin{aligned} \sigma_{\lambda'}(\tilde{z}') &= \sigma_{\lambda'}(z') + \frac{L'}{\lambda'_n} (z'_n - \beta(z')) \\ &= \left(\frac{L'}{\ell} - \frac{L'}{\lambda'_n} \right) \beta(z') \\ &\geq 0 \end{aligned}$$

Here we used that $\lambda'_n = \lambda_n + \ell > \ell$. It follows that $\tilde{z}' \in \overline{Q}(\lambda')$.

Set $\tilde{z} := \alpha^{-1}(\tilde{z}')$. By construction, $\tilde{z}_i = 0$ and $\tilde{z} \in \overline{Q}(\lambda)$. But we assumed that $Q(\lambda)$ has no holes in its boundary, thus $\tilde{z} \in Q(\lambda)$. It follows that $\tilde{z}' = \alpha(\tilde{z}) \in \alpha(Q(\lambda)) \subset Q(\lambda')$. But now [Lemma 3.3](#) implies that $z' \in Q(\lambda')$, a contradiction. \square

4. GOOD TRIPELS

In this section, we present our choice of the parameters λ . First, we show that for 3-dimensional rectangular simplices the hypothesis of [Proposition 3.1](#) is always satisfied.

Lemma 4.1. *A 3-dimensional rectangular simplex $Q(\lambda_1, \lambda_2, \lambda_3)$ has no holes in its boundary.*

Proof. The facets are 2-dimensional polytope affine semigroups. Thus, they are normal and even integrally closed in the ambient lattice \mathbb{Z}^4 (cf. [\[2, Corollary 2.54\]](#)). Hence, $Q(\lambda_1, \lambda_2, \lambda_3)$ has no holes in its boundary. \square

It is now sufficient to find $(\lambda_1, \lambda_2, \lambda_3)$ such that the distance of the holes to the facet F_λ is bounded below. This is archived with the following class of triples.

Definition 4.2. Let $\lambda_1 \leq \lambda_2 \leq \lambda_3$ be positive integers and let $\delta := (-1, 2, -1, 0) \in \mathbb{Z}^4$. We call $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ a *good triple* if the following conditions are met:

- (1) λ_1, λ_2 and λ_3 are pairwise coprime;
- (2) $\sigma_\lambda(\delta) = 2$, i.e. $\lambda_2\lambda_3 - 2\lambda_1\lambda_3 + \lambda_1\lambda_2 = 2$;
- (3) $\lambda_1 + 2 < \lambda_2$.

The following can be verified directly.

Proposition 4.3. *Let $\lambda_1 \geq 5$ be an odd positive integer. Then $(\lambda_1, 2\lambda_1 - 1, 2\lambda_1^2 - \lambda_1 - 2)$ is a good triple.*

Next, we show that good triplets yield indeed examples of simplices satisfying our need. So the next proposition completes the proof of [Theorem 1.1](#).

Proposition 4.4. *Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be a good triple. Then $Q(\lambda)$ is not normal and every hole has lattice distance at least $\lambda_1 + 2$ over F_λ .*

We prepare two lemmata before we prove this proposition.

Lemma 4.5. *Let $\lambda_1, \dots, \lambda_n$ be pairwise coprime. For every positive integer $s > 0$, there exists at most one element $\mathbf{q} \in \overline{Q}(\lambda)$ with $\sigma_\lambda(\mathbf{q}) = s$ and $\sigma_i(\mathbf{q}) < \lambda_i$ for every $1 \leq i \leq n$.*

Proof. This follows easily from the observation that $\ker \sigma_\lambda$ is generated as a group by $\mathbf{v}_1, \dots, \mathbf{v}_n$. The reasoning is inspired by the proof of Proposition 1.3 in [\[1\]](#). \square

Lemma 4.6. *Let $\lambda_1, \dots, \lambda_n$ be pairwise coprime and let s be a positive integer. If for every positive integer $t \leq s$, there exists an element $\mathbf{p}_t \in Q(\lambda)$ with $\sigma_\lambda(\mathbf{p}_t) = t$, then every hole $\mathbf{q} \in \overline{Q}(\lambda) \setminus Q(\lambda)$ has $\sigma_\lambda(\mathbf{q}) > s$.*

Proof. We may assume that $\sigma_i(\mathbf{p}_t) < \lambda_i$ and $\sigma_i(\mathbf{q}) < \lambda_i$ for every t and every i , because otherwise we can subtract \mathbf{v}_i . Now the claim is immediate from the preceding [Lemma 4.5](#). \square

Proof of Proposition 4.4. First, note that both λ_1 and λ_3 are necessarily odd. For this assume to the contrary that $\lambda_1 = 2\lambda'_1$ for an integer λ'_1 . Then $\lambda_2\lambda_3 = 2(1 + 2\lambda'_1\lambda_3 - \lambda'_1\lambda_2)$, thus either λ_2 or λ_3 are even, violating the coprimeness assumption. The proof that λ_3 is odd is analogous. Next, consider the vector

$$\begin{aligned} \mathbf{p} &:= \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_3 + \boldsymbol{\delta}) \\ &= \left(\frac{\lambda_1 - 1}{2}, 1, \frac{\lambda_3 - 1}{2}, 1 \right). \end{aligned}$$

It is easy to see that $\mathbf{p} \in Q(\boldsymbol{\lambda})$ and that $\sigma_{\boldsymbol{\lambda}}(\mathbf{p}) = 1$. For $0 \leq k \leq \frac{\lambda_1 - 1}{2}$ it holds that $\mathbf{p} + k\boldsymbol{\delta} \in Q(\boldsymbol{\lambda})$ and $\sigma_{\boldsymbol{\lambda}}(\mathbf{p} + k\boldsymbol{\delta}) = 1 + 2k$. Moreover, $2\mathbf{p} + k\boldsymbol{\delta} \in Q(\boldsymbol{\lambda})$ and $\sigma_{\boldsymbol{\lambda}}(2\mathbf{p} + k\boldsymbol{\delta}) = 2 + 2k$. Thus, by Lemma 4.6, there exists no hole with lattice height less than $\lambda_1 + 2$ above $F_{\boldsymbol{\lambda}}$.

Let

$$\begin{aligned} \mathbf{q} &:= \mathbf{p} + \left(\frac{\lambda_1 - 1}{2} + 1 \right) \boldsymbol{\delta} + \mathbf{v}_1 \\ &= (\lambda_1 - 1, \lambda_1 + 2, \frac{\lambda_3 - \lambda_1}{2} - 1, 2). \end{aligned}$$

The components of \mathbf{q} are non-negative and $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) = \lambda_1 + 2$, hence $\mathbf{q} \in \overline{Q}(\boldsymbol{\lambda})$. We claim that $\mathbf{q} \notin Q(\boldsymbol{\lambda})$, and thus that $Q(\boldsymbol{\lambda})$ is not normal. So assume that $\mathbf{q} = \mathbf{q}_1 + \mathbf{q}_2$ for $\mathbf{q}_1, \mathbf{q}_2 \in Q(\boldsymbol{\lambda})$. Since λ_1, λ_2 and λ_3 are pairwise coprime, the only elements of $Q(\boldsymbol{\lambda})$ in $F_{\boldsymbol{\lambda}}$ of degree 1 are $\mathbf{v}_1, \mathbf{v}_2$ and \mathbf{v}_3 . But $\lambda_1 - 1 < \lambda_1$, $\lambda_1 + 2 < \lambda_2$ (by assumption) and $\frac{\lambda_3 - \lambda_1}{2} - 1 < \lambda_3$, so $\mathbf{q} - \mathbf{v}_i$ has a negative component. It follows that $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1), \sigma_{\boldsymbol{\lambda}}(\mathbf{q}_2) > 0$. Since $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}) = \lambda_1 + 2$ is odd, one of $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1)$ and $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_2)$ is odd, too, say $\sigma_{\boldsymbol{\lambda}}(\mathbf{q}_1)$. By Lemma 4.5, all elements \mathbf{v} of $Q(\boldsymbol{\lambda})$ of degree 1 with $\sigma_{\boldsymbol{\lambda}}(\mathbf{v}) \leq \lambda_1$ odd are of the form $\mathbf{p} + k\boldsymbol{\delta}$ for $0 \leq k \leq \frac{\lambda_1 - 1}{2}$. But $\mathbf{q} - (\mathbf{p} + k\boldsymbol{\delta}) = \mathbf{v}_1 + (\frac{\lambda_1 - 1}{2} + 1 - k)\boldsymbol{\delta}$ has a negative third component. Thus \mathbf{q} cannot be written as a sum of elements of degree 1 in $Q(\boldsymbol{\lambda})$. \square

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